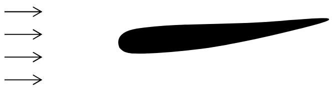
Boundary layer equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U_\infty\frac{dU_\infty}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}$$



Similarity Solution

- similarity variable: a general form $\eta(x,y)=yg(x)$
- similarity expect: $u(x,y)=U_\infty(x)f'(\eta)$
- transformed stream function:

$$\psi(x,y)=\int u\,dy+h(x)=\frac{U_\infty}{g}f(\eta)+h(x)$$

$$v(x,y)=-\frac{\partial\psi}{\partial x}=-\frac{d}{dx}\left(\frac{U_\infty}{g}\right)f-\frac{U_\infty}{g}f'yg'-h'$$

$$v(x,0)=v_0(x)=-\frac{d}{dx}\left(\frac{U_\infty}{g}\right)f(0)-h'(x)$$

$$h'(x)=-\frac{d}{dx}\left(\frac{U_\infty}{g}\right)f(0)-v_0(x)$$

Similarity Solution

$$v(x,y) = -\frac{d}{dx}\left(\frac{U_\infty}{g}\right) f - \frac{U_\infty}{g} f' y g' + \frac{d}{dx}\left(\frac{U_\infty}{g}\right) f(0) + v_0(x)$$

$$= v_0(x) - \frac{d}{dx}\left(\frac{U_\infty}{g}\right)\{f - f(0)\} - \frac{U_\infty}{g} f' y g'$$

$$u(x,y) = U_\infty(x) f'(\eta)$$

derivatives: $\frac{\partial u}{\partial x} = U_\infty' f' + U_\infty f'' \frac{\partial \eta}{\partial x} = U_\infty' f' + U_\infty f'' \cdot y g'$

$$\frac{\partial u}{\partial y} = U_\infty f'' \frac{\partial \eta}{\partial y} = U_\infty f'' \cdot g$$

X-momentum conservation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$v(x,y) = v_0(x) - \frac{d}{dx}\left(\frac{U_\infty}{g}\right)\cdot\{f(\eta) - f(0)\} - \frac{U_\infty}{g}\cdot f' \cdot y g'$$

$$u(x,y) = U_\infty(x) f'(\eta)$$

functions of x only

$$f''' = -\frac{1}{\nu g}\frac{d}{dx}\left(\frac{U_\infty}{g}\right)\cdot\{f - f(0)\} f'' + \left(\frac{dU_\infty/dx}{\nu g^2}\right)\left(f'^2 - 1\right) + \frac{v_0}{\nu g}\cdot f''$$

functions of η only

Consistence Condition

$$\frac{dU_\infty/dx}{\nu\, g^2} = \text{constant} = a$$

$$\frac{1}{\nu\, g}\frac{d}{dx}\left(\frac{U_\infty}{g}\right) = \text{constant} = b$$

$$\frac{v_0(x)}{\nu g(x)} = \text{constant} = B$$

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$$U_\infty(x) \propto x^{a/(2b-a)} \equiv Cx^m$$

$$g(x) \propto x^{(a-b)/(2b-a)} = Dx^{(m-1)/2}$$

$$\eta(x,y) = yg(x) = y\sqrt{\frac{U_\infty(x)}{\nu x}}$$

Consistence Condition

$$\frac{dU_\infty/dx}{\nu\, g^2} = \frac{m}{\nu}\frac{C}{D^2} \equiv m \qquad \therefore \quad D = \sqrt{C/\nu}$$

$$\frac{1}{\nu\, g}\frac{d}{dx}\left(\frac{U_\infty}{g}\right) = \frac{m+1}{2\nu}\frac{C}{D^2} = \frac{m+1}{2}$$

$$g(x) = Dx^{(m-1)/2} = \sqrt{\frac{Cx^{(m-1)}}{\nu}} = \sqrt{\frac{Cx^m}{\nu x}} = \sqrt{\frac{U_\infty(x)}{\nu x}}$$

$$\frac{v_0(x)}{\nu g(x)} = B \Rightarrow v_0(x) = B\nu g(x) \equiv B\sqrt{\frac{\nu U_\infty}{x}}$$

$$B \equiv \frac{v_0(x)}{\sqrt{\nu U_\infty/x}}$$

Blowing/suction condition

$$h'(x) = -\frac{d}{dx}\left(\frac{U_\infty}{g}\right) f(0) - v_0(x) \quad = \left\{-\frac{m+1}{2} f(0) - B\right\}\sqrt{\nu C x^{m-1}}$$

$$h(x) = \frac{2}{m+1}\left\{-\frac{m+1}{2} f(0) - B\right\}\sqrt{\nu C x^{m+1}} = \frac{2}{m+1}\left\{-\frac{m+1}{2} f(0) - B\right\}\sqrt{\nu x U_\infty}$$

$$\psi(x,y) = \frac{U_\infty}{g} f(\eta) + h(x) = \sqrt{\nu x U_\infty} f(\eta) + \frac{2}{m+1}\left(-\frac{m+1}{2} f(0) - B\right)\sqrt{\nu x U_\infty}$$

$$= \sqrt{\nu x U_\infty}\left\{f(\eta) - \frac{2}{m+1}\left(-\frac{m+1}{2} f(0) - B\right)\right\} \quad = \sqrt{\nu x U_\infty} f(\eta)$$

choose (without loss of generality) $-\frac{m+1}{2} f(0) \equiv B$

Remark

A similarity solution

$$\frac{dU_\infty/dx}{\nu g^2} = m$$

$$\frac{1}{\nu g}\frac{d}{dx}\left(\frac{U_\infty}{g}\right) = \frac{m+1}{2}$$

$$\frac{v_0(x)}{\nu g(x)} = B$$

$$\psi(x,y) = \sqrt{\nu x U_\infty(x)} \cdot f(\eta)$$

$$u(x,y) = U_\infty(x) f'(\eta)$$

$$v(x,y) = -\sqrt{\frac{\nu}{x} U_\infty(x)} \cdot \left\{\frac{m+1}{2} f(\eta) + \frac{(m-1)}{2} \cdot \eta f'(\eta)\right\}$$

$$\eta(x,y) = y g(x) = y\sqrt{\frac{U_\infty(x)}{\nu x}}$$

exists if $U_\infty(x) \propto x^m$ and $v_0 \propto x^{(m-1)/2}$

Falkner-Skan Equation

$$\underline{\underline{f'''}} = -\frac{1}{\nu\, g}\frac{d}{dx}\left(\frac{U_\infty}{g}\right)\cdot \underline{\underline{\{f - f(0)\}\, f''}} + \left(\frac{dU_\infty/dx}{\nu\, g^2}\right)\underline{\underline{\left(f'^2 - 1\right)}} + \frac{v_0}{\nu g}\cdot \underline{\underline{f''}}$$

$$2f''' + (m+1) f\, f'' + 2m\left(1 - f'^2\right) = 0$$

$$f'(0) = 0$$

$$f'(\infty) = 1$$

$$-\frac{m+1}{2}\cdot f(0) = B \equiv \frac{v_0(x)}{\sqrt{\nu U_\infty / x}}$$

$$u(x,0) = 0$$

$$u(x,\infty) = U_\infty(x)$$

$$v(x,0) = v_0(x)$$

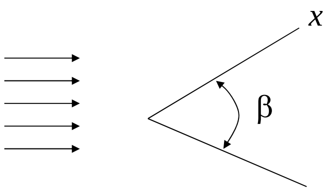
$$u(x,y) = U_\infty(x)\, f'(\eta)$$

$$-\frac{m+1}{2} f(0) = B = \frac{v_0(x)}{\sqrt{\nu U_\infty / x}}$$

Wedge Flows

$$U_\infty(x) \quad \propto \quad x^m$$

$$\frac{dP}{dx} = -\rho U_\infty \frac{dU_\infty}{dx} \neq 0 \qquad \text{i.e. } \; m \neq 0$$



$$m = \frac{\beta}{2\pi - \beta}$$

Thermal Equation

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}$$

$$\Theta \equiv \frac{T-T_0}{T_\infty - T_0}=\Theta(\eta) \qquad , \qquad \eta = y\sqrt{\frac{U_\infty(x)}{\nu x}}$$

$$T(y=0)=T_0(x)$$
$$T(y\to\infty)=T_\infty$$

$$n=\frac{x}{T_\infty - T_0}\frac{d(T_\infty - T_0)}{dx}$$

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$$\Theta'' + \frac{m+1}{2}\cdot \mathrm{Pr}\cdot f\,\Theta' + n\cdot \mathrm{Pr}\cdot(1-\Theta)\,f' = 0$$

$$\Theta(0)=0 \qquad \Theta(\infty)=1$$

Friction Coefficient

$$u(x,y)=U_\infty(x)\,f'(\eta)$$

$$\eta(x,y)=y\sqrt{\frac{U_\infty(x)}{\nu x}}$$

$$\tau_0=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}=\mu U_\infty\left(\frac{df'}{d\eta}\right)_{\eta=0}\left(\frac{\partial\eta}{\partial y}\right)_{y=0}=\rho\nu U_\infty f''(0)\sqrt{\frac{U_\infty}{\nu x}}$$

$$C_f=\frac{\tau_0}{\frac{1}{2}\rho U_\infty^2}=2f''(0)\sqrt{\frac{\nu}{U_\infty x}}=2f''(0)\,\mathrm{Re}_x^{-1/2}$$

$$C_f\,\mathrm{Re}_x^{1/2}=2f''(0)=\text{constant}$$

$$\mathrm{Re}_x\equiv U_\infty x/\nu$$

Nusselt Number

$$q''_0 = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = -k(T_\infty - T_0)\left(\frac{d\Theta}{d\eta}\right)_{\eta=0}\left(\frac{\partial \eta}{\partial y}\right)_{y=0} = -k(T_\infty - T_0)\Theta'(0)\sqrt{\frac{U_\infty}{\nu x}}$$

$$\Theta \equiv \frac{T - T_0}{T_\infty - T_0} = \Theta(\eta)$$

$$\eta = y\sqrt{\frac{U_\infty(x)}{\nu x}}$$

$$h = \frac{q''_0}{(T_0 - T_\infty)} = k\Theta'(0)\sqrt{\frac{U_\infty}{\nu x}} \quad \propto x^{(m-1)/2}$$

$$Nu = \frac{hx}{k} = \Theta'(0)\cdot\sqrt{\frac{U_\infty x}{\nu}} = \Theta'(0)\cdot \mathrm{Re}_x^{1/2}$$

$$Nu \cdot \mathrm{Re}_x^{-1/2} = \Theta'(0) = \text{constant (function of Pr)}$$

Pressure effect

$$\frac{dP}{dx} = -\rho U_\infty \frac{dU_\infty}{dx} \neq 0 \qquad \text{i.e.} \quad m \neq 0$$

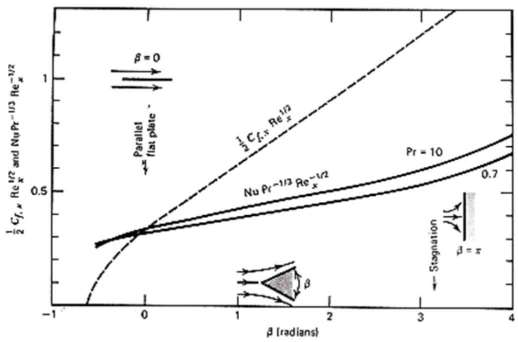


Figure 2.10 Overview of the local friction and heat transfer results for laminar boundary layer flow over an isothermal wedge-shaped body.

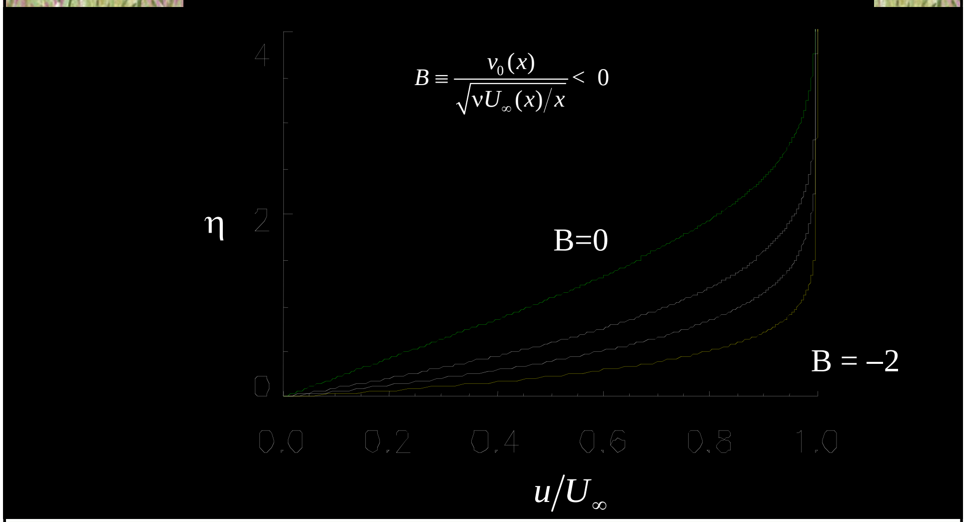
$$U_\infty(x) \propto x^m$$

$$m = \frac{\beta}{2\pi - \beta}$$

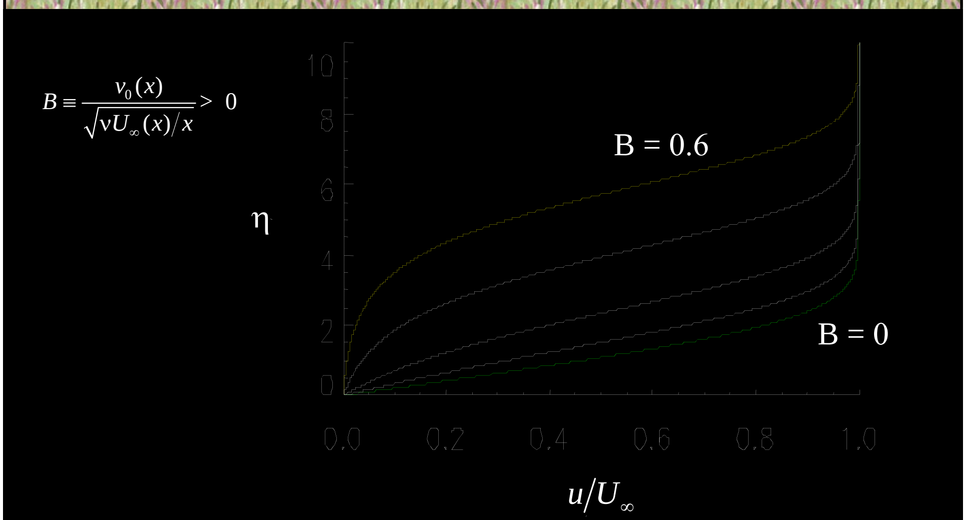
$$\tfrac{1}{2}C_f\,\mathrm{Re}_x^{1/2} = f''(0)$$

$$Nu\,\mathrm{Re}_x^{-1/2} = \Theta'(0)$$

Suction



Blowing



Blowing/Suction

blowing parameter:

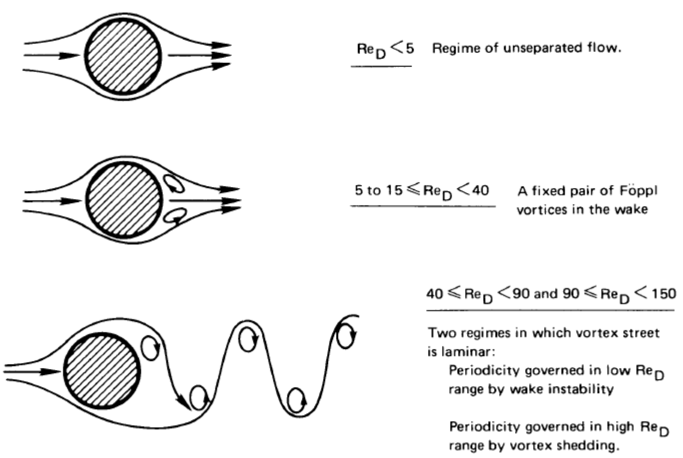
$$B \equiv \frac{v_0(x)}{\sqrt{\nu U_\infty(x)/x}} = -\frac{m+1}{2} \cdot f(0)$$

$$\tfrac{1}{2} C_f \, \mathrm{Re}_x^{1/2} = f''(0)$$

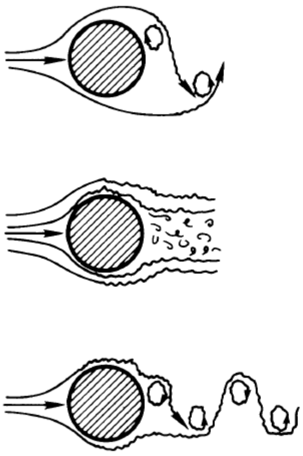
$$Nu \, \mathrm{Re}_x^{-1/2} = \Theta'(0)$$

		$\Theta'(0)$			
B	$f''(0)$	Pr = 0.7	Pr = 0.8	Pr = 0.9	
−2.5	2.59	1.85	2.097	2.59	Suction
−0.75	0.945	0.722	0.797	0.945	
−0.25	0.523	0.429	0.461	0.523	
0	**0.332**	**0.292**	**0.307**	**0.332**	impermeable
+0.25	0.165	0.166	0.166	0.165	blowing
+0.375	0.094	0.107	0.103	0.0937	
+0.5	0.036	0.0517	0.0458	0.0356	
+0.619	0	0	0	0	separation

Flow over a circular cylinder/sphere



Flow over a circular cylinder/sphere



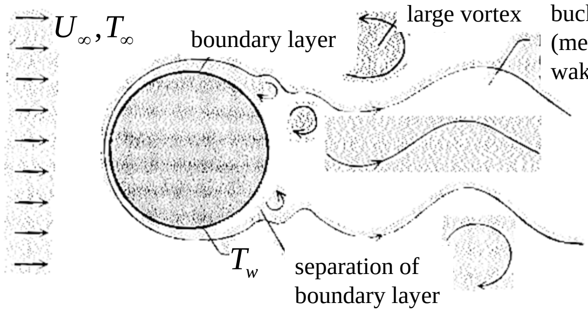
$150 \leqslant Re_D < 300$ Transition range to turbulence in vortex.

$300 \leqslant Re_D \lesssim 3 \times 10^5$ Vortex street is fully turbulent, and the flow field is increasingly 3-dimensional.

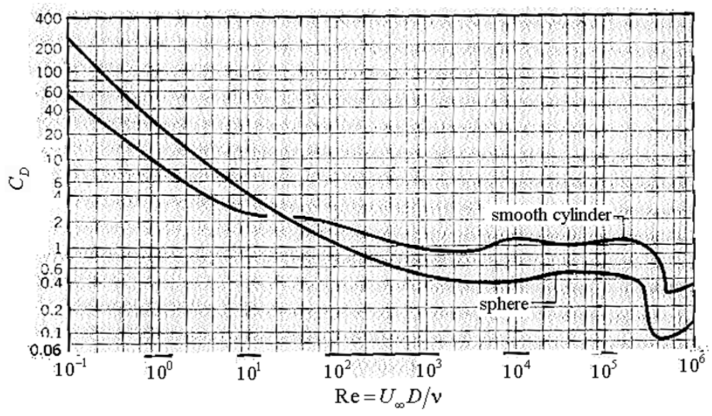
$3 \times 10^5 \lesssim Re_D < 3.5 \times 10^6$ Laminar boundary layer has undergone turbulent transition. The wake is narrower and disorganized. No vortex street is apparent.

$3.5 \times 10^6 \leqslant Re_D < \infty$ (?) Re-establishment of the turbulent vortex street that was evident in $300 \leqslant Re_D \lesssim 3 \times 10^5$. This time the boundary layer is turbulent and the wake is thinner.

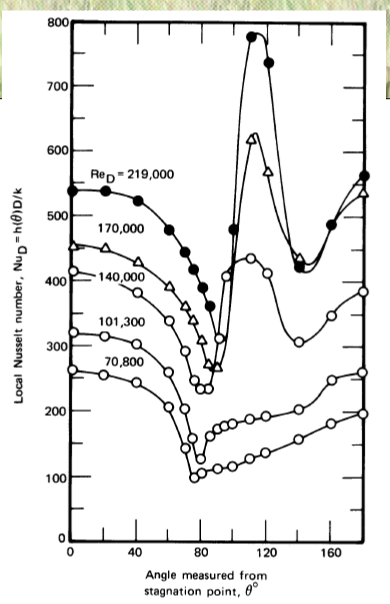
Single cylinder (or sphere) in cross-flow



Drag coefficient for smooth circular cylinder/sphere



Local Nusselt number for airflow normal to a circular cylinder



- Reynolds-number effect
- boundary layer
- laminar/turbulent
- separation (wake)